THE DEVELOPMENT OF ARITHMETIC CONCEPTS AND SKILLS IN SLOW LEARNERS

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A study of how slow learners learn arithmetic was completed after three months of extending remediation to nine grade 2 students from a Quezon City school. An analysis of arithmetic concepts and skills, as well as of the students' learning difficulties, yielded learning sequences that served as guides for remediation. These sequences were validated by analyzing learning behaviors and learning outcomes in the course of remediation. Observed learning processes elaborate on how children undergo the transition from the concrete mode to the symbolic mode of doing arithmetic. The major implication of the study is that a child's arithmetic thinking is essentially different from an adult's. That children know and think differently from adults should be an indication not so much of children's learning difficulties, as of the meanings of concepts that they can handle with ease and competence.

Introduction

Mathematics has long been regarded as a difficult subject. The almost universal perception of mathematics as an unwieldy field is shared not only by students from grade school to college, but by mathematics teachers as well. Certainly every mathematics teacher has grappled with various unsuccessful episodes in the classroom. Why do students who have, in other fields, exhibited the capacity for clear and logical thinking, fail to understand a seemingly simple mathematical concept? Indeed a student need not be learning disabled, or wanting in intelligence, for him to fail his school mathematics.

What have been written about mathematics learning? Literature on the learning of mathematics can be classified, according to content, into two: a) literature on the nature of mathematical structures, and b) literature on the teaching of mathematics.

Literature on the nature of mathematical structures describe and analyze mathematics as an axiomatic system (Baron, 1972; Mercer, 1972) and as a logical structure (Lovell, 1971). Emphasis is given on the requisites for understanding mathematics (Beard, 1972; Henkin, 1972; Plumpton, 1972).

Literature of the second type prescribe ways by which mathematics can be taught more effectively. Most of these prescriptions were derived from analyses of the nature of mathematics (Skemp, 1973). Some prescriptions, however, were based on analyses of how learners perceive what is taught to them. Authors note the importance of looking into learners' intuitive knowledge of mathematics, as well as their thinking processes and reasoning skills (Lovell, 1979; Novak, 1979; Oliver, 1972; Thijs, 1988).

Prescriptions for the teaching of mathematics can be categorized into two: a) prescriptions based on some theory, and b) prescriptions that the author, usually a teacher of mathematics, has found very useful and effective. Most theory-based prescriptions have yet to be tried out and validated. Prescriptions of the second type are usually specific and practical guidelines on teaching and can be applied directly to classroom situations. *Arithmetic Teacher* contains several of these.

Commentaries and reports have been made on the various problems encountered in teaching mathematics, especially in the primary and secondary levels. The Cockcroft Committee report (1982) on the teaching of mathematics has so far been the most comprehensive. Most of these commentaries and reports focus on the difficulties of most learners in understanding mathematics, as well as the ineffectiveness of several mathematics curricula in addressing the learners' needs. (Carraher, 1986; Cockcroft, 1982; Jaji, 1988; Williams, 1972). As a response, teaching strategies, which directly address students' learning difficulties, have been designed (Macnab and Cummine, 1986; Martin, 1986; Scott, 1972; Thijs, 1988).

Research in the learning of mathematics in the classroom have been undertaken. Soviet psychologists have pioneered in research in the psychology of teaching and learning mathematics. In these studies, mathematical structures, as well as students' thinking processes, were analyzed (Davydov, 1975a, 1975b; Gal'perin and Georgiev, 1977; Menchinskaya, 1977; Minskaya, 1975; Zykova, 1975, 1977). These studies deviated from the customary method of conducting correlational and experimental studies to determine and predict various trends in a population of students. Although correlational and experimental studies have worth of their own, they are unable to shed light on the processes involved in learning and teaching mathematics. In contrast, Soviet studies are able to shed light on these processes through observations of only a handful of students, but for prolonged periods of time.

Others have also realized the importance of focusing on processes within an individual, instead of trends within a population. Clinical methods of inquiry (Hughes, 1979), interview-about-instances approach (Osborne and Gilbert, 19[illegible]9), and classroom-based research with constraints removed (Delacote, 1979) are some methods that emphasize thinking processes.

In an attempt to understand and address arithmetic learning difficulties, we had, in this study, assumed the perspective of both teacher and learner. Much like an arithmetic teacher preparing for his lessons, we inquired into the nature of arithmetic structures and arithmetic teaching. We had gone beyond this task, however, and probed into children's minds, short of assuming their way of thinking. How exactly does a child's arithmetic thinking proceed? How does he make sense of numbers and operations? What concepts hold meaning for him, and what do not?

Rationale

In this study, we wanted to know how slow learners develop an understanding of the Decimal Numeration System and the operations of addition, subtraction, multiplication, and division.

First, we wanted to know how arithmetic learning difficulties come about. Since arithmetic learning difficulties are compounded, difficulties in learning higher level concepts and skills can be traced to an inadequate understanding of lower level concepts and skills. By interacting with slow learners, who were just beginning to learn arithmetic, we were able to trace the sources of difficulties in the learning of higher level concepts and skills.

Second, after studying the nature of learning difficulties, we wanted to know how arithmetic learning should proceed. We designed learning sequences (Swenson, 1983) that can serve as guides for the appropriate learning of arithmetic.

Third, we wanted to validate the learning sequences we had designed, that is, we wanted to know if indeed learning can proceed as indicated by these learning sequences. We taught arithmetic to slow learners using these learning sequences as guides, and we documented their learning behaviors and learning outcomes (Biggs and Collis, 1982).

Therefore, the primary premise of our study is that by interacting with slow learners, and by characterizing and addressing their learning difficulties, we can better understand how arithmetic learning proceeds.

This study did not include interactions with average and fast learners, because it is difficult to observe how these learners think. Average and fast learners readily, and sometimes spontaneously, learn arithmetic. Without prior detailed documentation of slower learning processes, it is difficult to keep track of faster learning processes. It is assumed, however, that observations generated in this study can eventually be used in studying arithmetic learning in average and fast learners.

Research Model

The research model we adapted assumes that arithmetic learning is hierarchical. When we say that arithmetic learning proceeds hierarchically, we mean that certain concepts and skills must be understood before more complex ones can be learned. If prerequisite concepts and skills were poorly learned, or not learned at all, then, the learning of succeeding concepts and skills would be difficult.

The above assumption necessitated two research tasks:

First, since we assumed that arithmetic learning proceeds in an orderly sequence, then, in order to determine such a sequence, it was necessary to analyze how arithmetic concepts and skills are related to each other. Our analysis of arithmetic concepts and skills generated building blocks of learning sequences.

Second, since we assumed that poorly learned concepts and skills give rise to difficulties in learning succeeding concepts and skills, then, in order to analyze a child's learning difficulties, it was necessary to situate the child's **level of understanding** (Brownell, 1987) in a hierarchy of arithmetic concepts and skills. Our analysis of concepts and skills was used in constructing an arithmetic assessment instrument.

The research model below served as framework of research procedures. Since this model was adapted twice: first, during a pilot phase; and, second, during the formal data gathering phase, the study, in effect, involved a two-stage validation of learning sequences. By including a pilot phase, we were able to refine research procedures before undertaking formal research. We were able to determine the content of learning sequences and assessment instrument, generate preliminary characterizations of learning difficulties, and improve the design of the remedial program.

A total of 252.5 hours, over a three-month period, was spent interacting with grade 2 students from a Quezon City school. In the pilot phase, remediation was conducted for a group of four children. In the formal data gathering phase, remediation was conducted for two groups of three children each. A child could not keep up with group teaching and was given individualized instruction.

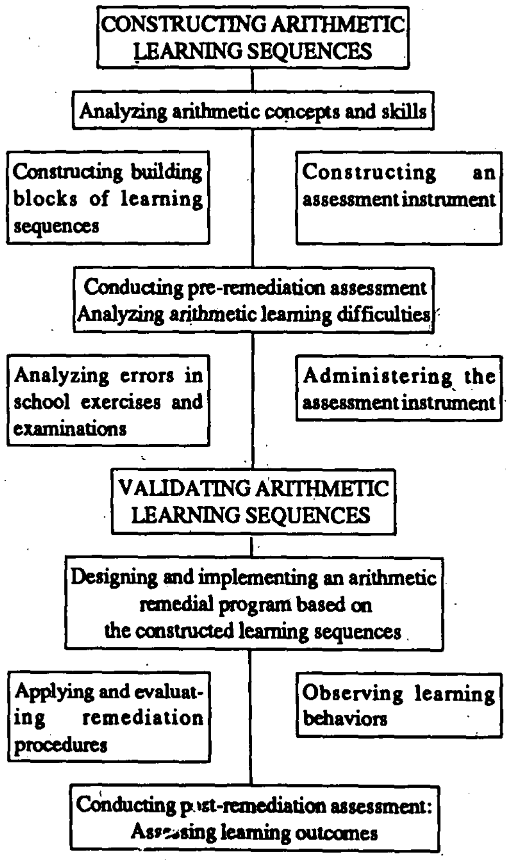

Fig. 1. Research model adapted for the study.

A detailed discussion of the sample of subjects, as well as of research procedures, appears in a separate volume.[1] Also contained in this volume are 49 tables documenting building blocks of learning sequences, learning difficulties, remediation procedures, and learning outcomes.

Conclusions: Validation of Arithmetic Learning Sequences

In analyzing arithmetic concepts and skills, we realized that certain lessons, which were not included in the children's[2] school exercises and examinations, should be included in the remedial program. We thought that the omission of these lessons forced the children to undergo abrupt transitions from lower to higher levels of understanding. Such transitions can be made less abrupt by tapping levels intermediate to higher levels of understanding. The intermediate levels incorporated in the remedial program are those that allow smooth transitions a) from manipulating objects to invoking relationships between operations, when operating on single-digit numbers,[3] and b) from operating on single-digit numbers to operating on multidigit numbers.

In analyzing arithmetic concepts and skills, we realized, too, that certain lessons in the children's school exercises and examinations should not be included in the remedial program. These lessons utilize symbols and manipulations of symbols that the children found difficult to understand. Documentations of the children's learning difficulties show that the symbolic mode of doing arithmetic is a major source of learning difficulties.

To further characterize intermediate levels of understanding and modes of doing arithmetic, we analyzed the children's learning difficulties as reflected in their performance in school exercises and examinations. These difficulties can be classified into three: a) difficulty in understanding the language used in teaching arithmetic, b) difficulty in comprehending and manipulating arithmetic symbols, and c) difficulty in coordinating the use of several concepts and skills.

We also analyzed learning difficulties as indicated by the children's performance in the assessment instrument and observed during the remedial program. Our analysis suggests two learning outcomes that the children could not quite achieve, namely: a) to develop meanings of arithmetic concepts in the concrete mode, and b) to translate to the symbolic mode the meanings of concepts previously learned in the concrete mode.

We realized that the minimal use of the concrete mode in school arithmetic could have resulted in the children's difficulty in comprehending and manipulating symbols. We observed that children who had difficulty in comprehending and manipulating symbols also had difficulty in coordinating the use of several concepts and skills. This is so because most tasks require a coordinated use of concepts and skills in the symbolic mode.

Therefore, for the remedial program, we decided to allow the children to rely heavily on the concrete mode, specially at the lower and intermediate levels of understanding. At the initial levels of working in the symbolic mode, the children would also have to refer back to the concrete.

It is often necessary for children to describe what they do with concrete tools. We therefore used language as a mode supplementary to the concrete. We used Filipino because it is the language in which the children could best explain their ideas. Because most of the children had reading difficulties as well as difficulties in understanding arithmetic terms, we minimized the use of written and technical language, and resorted instead to spoken and informal language.

In summary, we categorize remediation procedures into three: a) facilitate and encourage com-

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[1] Lopez, Melissa Lucia J. The Development of Arithmetic Concepts and Skills in Slow Learners. Unpublished master's thesis, University of the Philippines, 1991.

[2] The term "the children" is used to refer to the participants of the study.

[3] Operating on numbers means adding, subtracting, multiplying, and dividing them.

munication between teacher and learner, and among fellow learners, b) develop children's understanding of concepts in the concrete mode before translating the meanings of these concepts to the symbolic mode, and c) help children direct and guide their own thinking so they can better coordinate the use of several concepts and skills.

These categories of procedures are interrelated. Improved communication between teacher and learner, and among fellow learners, facilitates the learning of arithmetic in the concrete mode. Moreover, the ability to communicate with others in the course of learning, and an adequate exposure to the concrete mode should enable children to direct and guide their own thinking as they begin to work in the symbolic mode.

It should be emphasized that the concrete mode includes not only manipulations of objects, but also counting processes, real-life situations, and even spoken and informal language. These are the tools used by the children in working toward the desired learning outcomes. Utilizing counting processes to explore arithmetic concepts and skills constitutes the intermediate levels of understanding that the children underwent. As the children progressed to higher levels of understanding, they were beginning to work more efficiently in the symbolic. Gradually, concrete tools were replaced by number sentences and written numerical solutions, with the latter being specially useful in developing the ability to operate on multidigit numbers.

In summary, the arithmetic learning processes observed during the remedial program are as follows:

1. The children first learned of how groups of objects are symbolized under the Decimal Numeration System when they began to perceive numbers as made up of 100s, 10s, and 1s; and when they began to perceive a group of 100 or a group of 10 as a single entity similar to a group of 1.
2. The children learned various strategies for operating on single-digit numbers. The development of these strategies is a result of the ability to perceive operations in terms of increasingly symbolic processes. First, the children manipulated objects to operate on single-digit numbers. Second, the children substituted counting processes for manipulations of objects. Third, the children operated on single-digit numbers by invoking the operation's relationship to more basic operations.
3. The children learned to operate on multidigit numbers by adapting concepts and skills for operating on single-digit numbers. The children recognized that operating on multiples of 10 and 100 is a process similar to operating on single- digit numbers. An understanding of the standard procedures for operating on multidigit numbers was made possible by allowing children to encounter analogous procedures in the concrete mode.
4. As the children developed skills in solving word problems, they learned to better recognize arithmetic operations in real-life situations. As in their strategies for operating on single-digit numbers, the children's solutions to word problems were couched in terms of increasingly symbolic processes: first, in terms of manipulations of objects; then, in terms of counting processes; and finally, in terms of number sentences.

In conclusion, the remedial program was instrumental in prompting the children to direct their own arithmetic thinking. Initially, the children did arithmetic in the concrete. At the intermediate and higher levels of understanding, however, the children's arithmetic thinking was directed not so much by what the children could see and do in the concrete, but what they could think of through symbols. At these levels, arithmetic concepts could be thought of, and the symbols for these concepts manipulated, independent of the concepts' concrete manifestations. In effect, thinking through formalized symbolizations was beginning to replace concrete tools.

For illustrative purposes, we include in this article the validated learning sequence for the concept of number.

A Validated Learning Sequence for the Concept of Number

The transition from perceiving the counting unit 1 to perceiving other counting units, such as 10 and 100, is an important one. An appreciation of 10 and 100 as counting units enables a child to understand how groups of objects are symbolized under the Decimal Numeration System.

When 1 is used as counting unit, each object in a group is regarded as apart from the rest, and objects are counted one by one. Thus, the resulting counting sequence is: 1, 2, 3, and so on.

When 10 or 100 is used as counting unit, every 10 or 100 objects in a group is regarded as apart from the rest. A group of 10 or 100 is not only perceived as composed of 10 or 100 objects; it is also perceived as a single object or entity. The resulting sequence when 10 is used as counting unit is: 10, 20, 30, and so on. The counting sequence for 100 is: 100, 200, 300, and so on.

When a child undergoes Levels B and C of the learning sequence below, he realizes that counting units other than 1 exist. Yet, being aware of the existence of other counting units does not make a child automatically prefer them to the counting unit 1. 1 is the basic counting unit; exclusive of fractions, it can no longer be broken down into anything smaller. It is basic because it is the counting unit used in the simplest counting task: that of going through the sequence 1, 2, 3 ... A child will continue to use 1 as counting unit unless he is faced with situations that compel him to use other counting units.

It is only when a child operates on multiples of 10 and 100 (Level D) that he learns to spontaneously use other counting units. The child learns that operating on multiples of 10 and 100 is a process similar to operating on single-digit numbers. Hence, he obtains 20+30 and 200+300 in the way he obtained 2+3. He realizes that 2x30 and 2x300 are more easily obtained by noting their similarity to 2x3.

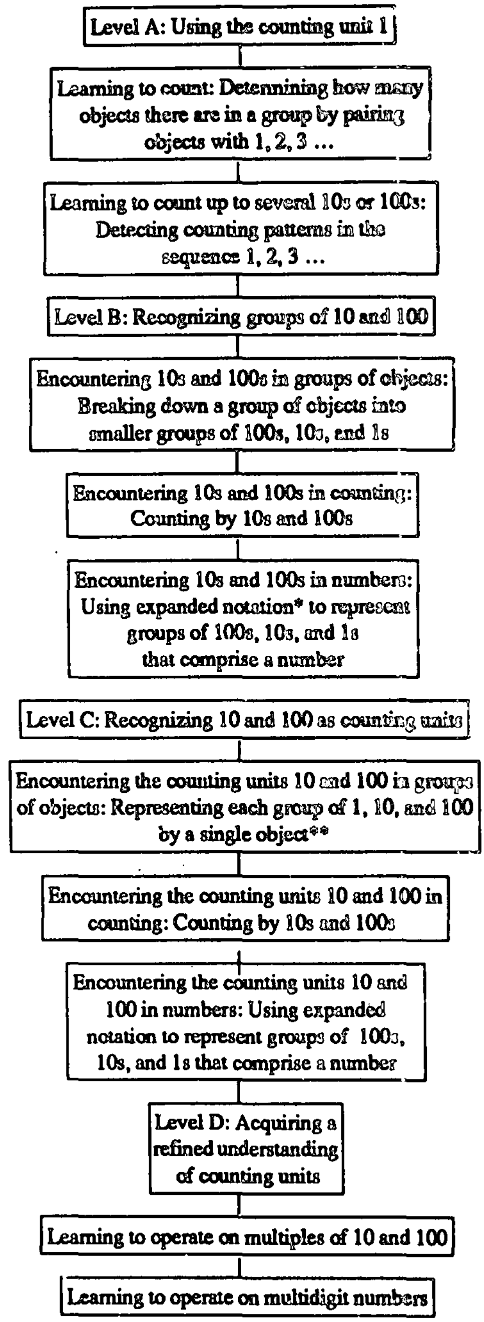

Fig. 2. A learning sequence for the concept of number

*In expanded notation, a number is recorded such that the 100s, 10s, and 1s which comprise it are made explicit. Hence, 234 in expanded notation is 200+30+4.

**Small objects can be used to represent 1s; medium-sized objects, to represent 10s; and large objects, to represent 100s.

Operating on multidigit numbers is another task that compels a child to perceive 10 and 100 as counting units. Operating on multidigit numbers entails decomposing them into 100s, 10s, and 1s. For example, in adding 234 and 123, numbers are decomposed, respectively, into 200, 30, 4, and 100, 20, 3. The sums of 200 and 100, 30 and 20, and 4 and 3 are then obtained and added. If a child cannot detect in 234 and 123, groups of 100, 10, and 1, then he will have to perceive 234 as two hundred thirty four 1s and 123 as one hundred twenty three 1s. Adding these numbers becomes complicated, because it is difficult to concretely represent or perceive a large group of 1s.

Operating on multidigit numbers also entails switching from one counting unit to another. Prior to learning operations of multidigit numbers, children learn to switch from one counting unit to another by regrouping in 10s, objects grouped in 1s; and by regrouping in 100s, objects grouped in 10s. Children also eventually learn the reverse of these regrouping tasks.

However, after children have mastered the above tasks (which can be incorporated in Levels B and C of the learning sequence), it is still likely that, when necessary, they will fail to switch from one counting unit to another. It is only when children operate on multidigit numbers that they begin to regroup more spontaneously. For note that regrouping is essentially the "carrying" process in addition and the "borrowing" process in subtraction. Regrouping processes are also involved in multiplying and dividing multidigit numbers. It becomes clear, therefore, that some difficulties in learning operations of multidigit numbers can be traced to an undeveloped understanding of the counting units 10 and 100.

To summarize, the general direction of the development of the concept of number is from recognizing groups of objects through counting to knowing how groups of objects are symbolized under the Decimal Numeration System. Through developed counting skills, children become aware of the existence of counting units other than 1. The spontaneous use of these counting units is achieved after children have learned to operate on multiples of 10 and 100 as well as other multidigit numbers.

Significance of the Study: What Have We Learned About Children's Arithmetic Thinking?

After observing how slow learners develop an understanding of arithmetic concepts and skills, we are inclined to believe that there exist differences between a child's and an adult's arithmetic thinking. These differences are essential: they lie not so much in the number of concepts and skills that are understood, as in the manner in which concepts and skills are understood. In conjecturing about such differences, we do not simply mean that adults know more, and children know less; what we do mean is that children know and think differently from adults.

Unfortunately, differences between children's and adults' arithmetic thinking are often overlooked in the design of arithmetic curricula and instructional materials. Even a cursory analysis of arithmetic textbooks suggests a commonplace tendency to package lessons not in a way that makes most sense to children, but in a way that makes most sense to adults. Several arithmetic concepts and skills are discussed from an adult's point of view, and often in so brief and superficial a manner that children can barely explore their ways of doing arithmetic before acquiring the ways of adults.

To illustrate how children's arithmetic thinking differs from adults', let us consider the following: How do children perceive multidigit numbers? How do children perceive arithmetic operations? What is the nature of children's arithmetic thinking?

How Do Children Perceive Multidigit Numbers?

We incorrectly assume that children's grasp of multidigit numbers is spontaneous, and are surprised whenever children have difficulty with them. Children find it difficult to perceive multidigit numbers because they insist on perceiving these numbers in the manner by which they learned to perceive single-digit numbers. Children tend to perceive multidigit numbers as made up of so many 1s, rather than as made up of so many 100s, 10s, and 1s. Perhaps the number 765 lies beyond the reach of most young children's minds—as 765 billion lies beyond the reach of most adults' minds. Whereas children can easily imagine 9 objects, it takes some long and tedious counting before they can display and visualize 765 objects.

Indeed, perceiving large numbers is a tedious, if not impossible task, were it not for our abilities to perceive groups of objects as single entities and, accordingly, to think in terms of counting units larger than 1. Every arithmetic teacher should realize that these abilities entail a certain level of abstraction and should not, therefore, be regarded as natural or spontaneous in children.

If we were to seriously consider the manner in which children perceive multidigit numbers, we would be disposed to restructure our method of teaching operations of multidigit numbers. Ordinarily, we teach children how to operate on multidigit numbers by giving them rules to memorize and apply. In obtaining 23 x 5, for example, children verbalize these rules as follows: 3x5=15; carry 1; 2x5=10; plus 1, equals 11; 23x5=115. Nowhere in this verbalization is it suggested that children have been taught to count five groups of 23 by constructing five groups of two 10s and five groups of three 1s.

There are lessons to be learned from operations of multidigit numbers which are more valuable than memorized rules for manipulating numbers. When operations of multidigit numbers are learned only in terms of memorized rules, children fail to see how very much related these rules are to the use of the counting units 10 and 100. They are also unable to realize that operations of multidigit numbers are essentially similar to operations of single- digit numbers.

It is to children's benefit to postpone—and in some cases, to altogether exclude—memorizing rules for operations of multidigit numbers. By placing less emphasis on memorized rules, we encourage children to explore arithmetic meanings on their own terms and at their own pace. Generally, operations of multidigit numbers are learned by encountering parallel procedures at increasingly symbolic levels: first, by experimenting on objects representing the counting units 1, 10, and 100; second, by working on numbers' expanded notation form; and, third, by applying rules for manipulating numbers. Each child should be allowed to seek his own level and to persist in doing arithmetic at this level, should he find it difficult to proceed to the next.

How Do Children Perceive Arithmetic Operations?

It should not be assumed that children have a natural and spontaneous grasp of arithmetic operations. Arithmetic operations should not be taught one after another, and in a short span of time, as if children are merely accumulating and memorizing facts. Just as perceiving multidigit numbers entails a restructuring of children's arithmetic thinking, so too does perceiving a new operation entail such restructuring.

Children attain an adequate understanding of the four arithmetic operations by acquiring increasingly sophisticated ways of perceiving numbers. By ways of perceiving numbers, we mean the manner by which the numbers recited in counting are paired with the objects being counted.

We learned of four ways by which children pair numbers with objects. A number can be paired with a single object (object <——> number), with a group of objects (group of objects <——> number), with two groups of objects (two groups of objects <——>number), or with a certain number of groups of equal sizes (groups of objects <——> number).

At the initial level of understanding, children's perception of the counting sequence 1, 2, 3 ... is limited, for numbers are thought of mainly in terms of the pair object<—>number. The counting sequence is perceived as made up of the pairs:

Objects in a group		Counting sequence
*	<—>	1
*	<—>	2
*	<—>	3
	⋮	
*	<—>	n
******...*	<—>	n.

1 is paired with the first object, 2 is paired with the second object, 3 is paired with the third object, and so on. The final count "n" is paired not only with the "n"th object, but also with the group of "n" objects. Thus, the pair group of objects<—>number is arrived at only at the end of the counting process.

At a higher level of understanding, children are able to think of numbers mainly in terms of the pair group of objects<—>number. That children pair a number with a group of objects rather than with an object, suggests a refinement in their perception of the counting sequence. The counting sequence is now perceived as made up of the pairs:

Objects in a group		Counting sequence
*	<—>	1
**	<—>	2
***	<—>	3
****	<—>	4
*****	<—>	5
******	<—>	6
	⋮	
****...*	<—>	n.

The counting sequence, conceptualized in this manner, becomes a source of number relationships, enabling children to arrive at addition facts without having to draw or manipulate objects. For instance, to obtain 8+5 from 7+5=12, children compare 7 and 8 through the following portion of the counting sequence:

*******	<—>	7
********	<—>	8

Knowing that 7+5=12, they can translate the above portion of the counting sequence to the following portion:

************	<—>	7 + 5 (12)
*************	<—>	8 + 5 (13),

to obtain 8+5=13. It would not have been possible to resort to this strategy if each of the numbers 7, 8, 12, and 13 were paired, not with a group of objects, but with a single object.

Thus, by invoking the pair group of objects-number, children can arrive at addition facts, without having to draw or manipulate objects. Similarly, they can obtain subtraction facts without having to manipulate objects by invoking the pair two groups of objects<—>number. Multiplication and division facts are obtained through the pair groups of objects<—>number. Consider, for instance, some of the strategies that children use:

Subtraction

Strategy: Invoking known sums

Example: To get 16–9, use 7+9=16 or 9+7=16

Pairs used: *group of 9 objects/group of 7 object<—>16:* directly established from a number sentence.

Multiplication

Strategy: Forming sets of addends

Example: To get 6x7: If three 7s is known, form two sets of three 7s. Get 21+21.

Pairs used: *three groups of 7<—>21:* forming each sets of three 7s
six groups of 7<—>42: getting 21+21=42

Division

Strategy: Invoking known products

Example: To get 42/6: Note that 5x6=30 and 2x6=12. Adding these products results in seven 6s, which is equal to 42. Thus 42/6=7.

Pairs used: *five groups of 6:<—>30;* two groups of 6<—>12: directly established from number sentences

seven groups of 6<—>42: adding 30 and 42

To assume that children spontaneously acquire strategies such as those above would be to assume that they can spontaneously generate the various pairs of numbers and objects. We learned, however, that children do not spontaneously generate these pairs. Instead, they utilize pairs that they have already mastered, to generate pairs that they have yet to spontaneously invoke. The counting processes that children resort to in generating complex pairs from basic ones constitute the transition from actual manipulations of objects to imagined manipulations of objects. Consider for instance some of the strategies that rely on such counting processes:

Addition

Strategy: Counting on

Example: To get 5+6: Either keep in mind or say aloud the number 5. Construct a group of 6 objects, while counting: "6, 7, 8, …,11." The last number in the count is 11. Therefore, 5+6=11.

Pairs used: *group of 5 objects<—>5*
group of 6 objects:
1st object<—>6
2nd object<—>7
3rd object<—>8
4th object<—>9
5th object<—>10
6th object<—>11
group of 6 objects<—>6
group of 11 objects<—>11

Subtraction

Strategy: Counting on

Example: To get 16–9: Count forward from 10 to 16. Determine the number of units counted.

Pairs used: *group of objects<—>number:* recognizing groups of 9, 16, and 7
object<—>number: counting forward from 10 to 16
group of 7 objects/group of 9 objects<—>16: recognizing 7 as 16–9

Multiplication:

Strategy: Using double counters

Example: To get 3x4:

Count	Left hand counter no of grps.		Right hand counter no. of objects in a grp.	
1			1	finger/s
2			2	
3			3	
4	1	finger/s	4	
5			5	
6			6	
7			7	
8	2		8	
9			9	

10		10
11		11
12	3	12

Pairs used: *object<——>1, object<——>2, object<——>3, object<——>4; group of 4 objects<——>4:* constructing each group of 4

one group of 4<——>4; two groups of 4<——>8; three groups of 4<——>12: determining the number of objects in three groups of 4

Division

Strategy: Skip counting

Example: To get 15/3: Count by 3s until 15 is reached. Five numbers are included in the count.

Pairs used: *group of 3 objects<——>3; one group of 3<——>3:* at "3"

group of 3 objects<——>3; two groups of 3<——>6: at "6"

group of 3 objects3; three group of 3<——>9: at "9"

:

:

group of 3 objects<——>3; five groups of 3<——>15: at "15"

Several other strategies are contained in the learning sequences we have designed and validated. Even without discussing these learning sequences however, it is clear from the above strategies that learning arithmetic operations entails more than just accumulating new knowledge and memorizing number facts. Children do not merely encounter an operation, and another, and still another. Learning operations entail a restructuring of one's perception of numbers. In this study, we had, on several occasions, detected children's potential to use various counting processes in their attempt to explore the more complex meanings of numbers.

Unfortunately, this potential is almost often left untapped. Some of the children we worked with had difficulty recalling the meanings of operations, despite their having received school lessons on the four operations. These children were equipped with the minimum of skills. They would painstakingly resort to drawing and manipulating objects to derive even the simplest number facts, such as 5+6 and 3x4.

Why do some children persist in drawing or manipulating objects to obtain even the simplest number facts? Why can they not derive these number facts "in their minds"? Even if children are not forced to memorize number facts, even if they are not hustled to blurt these out, children can refrain from drawing or manipulating objects—*but on their own terms*. For although children are motivated not to resort to manipulations of objects, they are not yet capable of arriving at number facts "in their minds," a skill we assume children possess whenever we subject them to speed drills and contests. When children no longer draw or manipulate objects, we should not even begin to think that they are simply imagining numbers. They do not; they count.

Our conjecture is that counting processes constitute the transition from actual manipulations of objects to imagined manipulations of objects. If this transition were overlooked, the necessary restructuring of children's arithmetic thinking would not be achieved.

What Is the Nature of Children's Arithmetic Thinking?

The major implication of our study is that a child's arithmetic thinking is essentially different from an adult's. The tendency to ignore children's perceptions of arithmetic concepts arises from the incorrect assumption that children's thinking is similar to adults. The need to restructure children's arithmetic thinking is hardly recognized.

In an attempt to illustrate differences between a child's and an adult's arithmetic thinking, we have discussed how children perceive multidigit numbers and arithmetic operations. In this section, we shall cite examples to show that whereas an adult's arithmetic thinking is deliberate and purposeful, a child's arithmetic thinking is random and exploratory.

More than one meaning can be attached to a number sentence. 16–9=7, for example, has three possible meanings: a) a group of 7 is what remains of a group of 16 after removing 9 objects from it (take away model); b) 7 objects must be added to a group of 9 to come up with a group of 16 (missing addend model); and, c) 7 more objects are in a group of 16 than in a group of 9 (comparison model).[4]

We tend to assume that children can shift from one meaning of subtraction to another. We learned however, that although children can readily attach the take away model to subtraction, they cannot do the same for the missing addend and comparison models. Children more readily attach the missing addend and comparison models to addition. They learn to associate these models with subtraction only after they have explored the relationship of subtraction to addition.

Furthermore, consider the following. If 16–9=7 is interpreted in terms of the take away model, then 9 and 7 are perceived as parts or components of 16 (group of 9 objects/group of 7 objects<——>16). If 16–9=7 is interpreted in terms of either the missing addend or comparison models, then 9 and 7 are perceived, not as parts or components of 16, but as entities separate and different from 16. Considering the different ways by which numbers are perceived to be related to one another, it should not come as a surprise to us that children fail to abstract a similarity among the three models of subtraction.

Children therefore have limited capacity to appreciate the different meanings of an operation. They also have limited capacity to choose which among an operation's various meanings is most useful in a given situation. Children tend not to choose the meaning which is most useful but that which they can most spontaneously understand. For example, to obtain the product 23x5, we are more inclined to work with five groups of 23 than with twenty three groups of 5, since the former meaning results in a shorter counting process. Some children however, would rather count by 5s twenty three times. For these children, counting by 5s is a natural skill, whereas counting by 23s is facilitated only if 23 is perceived in terms of the counting unit 10.

Children have difficulty too, in choosing the meaning of division that is most useful in a given situation. The meanings of division refer to ways of dividing objects into groups of equal sizes. The number sentence 20/5=4, for example, can refer to any of the following procedures: a) divide 20 into five groups of equal sizes, and b) divide 20 into groups of 5 objects each.

Dividing a multidigit number by a single-digit number is facilitated if the appropriate meaning of division is used. Children would find it difficult to get 700/7, if they were to think of 7 as group size and search for the number of groups. This procedure entails counting or adding 7s until 700 is reached. However, if children were to think of 7 as the number of groups, and of 700 as seven 100s, then they would easily recognize 100 as group size.

Most children detect which meaning of an operation is most appropriate in a given s[illegible] not through planned and deliberate [illegible] of an operation's various meanings, [illegible] a trial-and-error process of [illegible] counting objects. Oftentimes, child[illegible] ble across the desired meaning. [illegible] mentation, and not logic, is the basis of [illegible] arithmetic thinking.

[4] The terms "take away," "missing addend," and "comparison models" were adapted from Copeland (1982). Swenson (1983) refers to the missing addend as "additive model."

What Have We Learned About Children's Arithmetic Thinking?

In our search for the reasons why arithmetic learning difficulties are so pervasive among our children, we chanced upon a way of thinking that adults have probably outgrown, but which makes most sense to children. Although we assume that an adult's arithmetic thinking is superior to a child's, we do not mean to say that a child's way of perceiving arithmetic is of no value. It definitely has, if only because it is by assuming a child's perception, more than an adult's, that an arithmetic teacher can effectively communicate with his students.

That children perceive arithmetic differently from adults should be an indication not so much of children's learning difficulties, as of the meanings that they can handle with ease and competence. There were several occasions in the course of remediation when, in our attempt to analyze learning difficulties, we discovered children's untapped capacity to think, and to think well, *but on their own terms*. Could it be that several of our children's arithmetic learning difficulties are mere artifacts of our unreflected and unrealistic expectations of what they are capable of doing?

As we allow children to think of arithmetic concepts in their own terms, let us also allow them to talk about what they think in their own language. Written and technical language should be dispensed with in favor of spoken and informal language. How else can we probe into children's minds if not by adapting their language? More crucial than probing into children's minds however, is encouraging children to talk about their solutions to arithmetic tasks. When a child is encouraged to talk, she becomes more aware of what she is thinking of, begins to plan her actions and, consequently, gains purposefulness in her thinking. In turn, as a child learns to make purposeful decisions on what to think about and how to think about them, her arithmetic thinking becomes sophisticated (shall we say, much like an adult's), and she learns to couch the purposeful decisions she makes in terms of the symbolic.

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